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Orthomodular Lattices of Subspaces Obtained from Quadratic Forms

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Being given a field K of characteristic different from 2 and 3, a 3-dimensional vector space E over K, and a nonsingular symmetric bilinear form φ over E, we define a structure of orthomodular lattice $T(E, \varphi)$ on the set of all nonisotropic subspaces of E.

We give a structure Theorem about the irreducible and 3-homogeneous subalgebras of $T(E, \varphi)$. In particular, these subalgebras are all of the form $T(E', \varphi')$ where E' is a 3-dimensional subspace of E, if E is regarded as a vector space over a subfield K' of K, and φ' is induced by φ.

This structure Theorem allows us to achieve an old project, concerning minimal orthomodular lattices (an orthomodular lattice L is called minimal if it is nonmodular and if all its proper subalgebras are either modular, or isomorphic to L).

KEY WORDS: Orthomodular lattice; quadratic space; polarity; variety.

1. THE MODULAR LATTICE $L(E, \varphi)$

Let K be any field of characteristic different from 2 and 3.

Let E be a 3-dimensional vector space over K.

Let $\varphi : E \times E \to K$ be a non singular symmetric bilinear form and $Q : E \to K$ the quadratic form associated to φ.

Two vectors u, v in E are said to be φ-orthogonal, which is denoted by $u \perp v$, if $\varphi(u, v) = 0$. For any subspace M of E, the set $\{u \in E | \forall v \in M, u \perp v\}$ is a subspace of E denoted by $M^\perp$.

We denote by $L(E, \varphi)$ the modular lattice of all subspaces of E equipped with the map $M \mapsto M^\perp$.

The elements of $L(E, \varphi)$ are $\{0\}$, E, the 1-dimensional subspaces Ku (atoms of $L(E, \varphi)$), and the 2-dimensional subspaces $(Ku)^\perp$ (co-atoms of $L(E, \varphi)$).

The modular lattice $L(E, \varphi)$ is a projective plane, and the map $M \mapsto M^\perp$ is the polarity with respect to a conic.

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The lattice operations on $L(E, \varphi)$ are defined by:

$$Sup_L(M_1, M_2) = M_1 + M_2$$
$$Inf_L(M_1, M_2) = M_1 \cap M_2$$

The polarity $M \mapsto M^\perp$ is (by definition) involutive and decreasing. It follows that the de Morgan laws are satisfied in $L(E, \varphi)$:

$$(M_1 + M_2)^\perp = M_1^\perp \cap M_2^\perp$$
$$(M_1 \cap M_2)^\perp = M_1^\perp + M_2^\perp$$

2. THE ORTHOMODULAR LATTICE $T(E, \varphi)$

In the general case, the polarity $M \mapsto M^\perp$ is not an orthocomplementation on the lattice $L(E, \varphi)$, since some subspaces M can be isotropic. Let us remind the definition of an isotropic subspace.

A nonzero vector $u \in E$ is called isotropic if $Q(u) = 0$.

A subspace $M \in L(E, \varphi)$ is called isotropic if it is nonzero and the restriction of φ to $M \times M$ is singular, that is to say if $M \cap M^\perp \neq \{0\}$.

Actually, a subspace M of $L(E, \varphi)$ is isotropic if either it is of the form $K\omega$ where ω is an isotropic vector, or of the form $(K\omega)^\perp$ where ω is isotropic.

We note that if M is an atom (resp. a co-atom) of $L(E, \varphi)$, then M is isotropic if and only if $M \subseteq M^\perp$ (resp. $M^\perp \subseteq M$).

Let us denote by $T(E, \varphi)$ the set of all nonisotropic subspaces of E. In the general case, the set $T(E, \varphi)$ is not a sublattice of $L(E, \varphi)$. However, when ordered by inclusion, it is a lattice whose operations are defined as follows :

$$M_1 \vee M_2 = \begin{cases} M_1 + M_2 & \text{if} \quad M_1 + M_2 \quad \text{is nonisotropic} \\ E & \text{if} \quad M_1 + M_2 \quad \text{is isotropic} \end{cases}$$

$$M_1 \wedge M_2 = \begin{cases} M_1 \cap M_2 & \text{if} \quad M_1 \cap M_2 \quad \text{is nonisotropic} \\ \{0\} & \text{if} \quad M_1 \cap M_2 \quad \text{is isotropic} \end{cases}$$

Moreover, the map $M \mapsto M^\perp$ is an orthocomplementation on $T(E, \varphi)$, and, in particular, the de Morgan laws are satisfied in $T(E, \varphi)$.

Lemma 2.1.

1. *Each plane in $L(E, \varphi)$ contains at least six atoms and contains at most two isotropic atoms.*
2. *Each nonisotropic atom of $L(E, \varphi)$ is contained at least in four non-isotropic planes.*

Proof:

1. Since the characteristic of K is different from 2 and 3, the cardinality of K is at least 5, hence each 2-dimensional subspace of E contains at least six 1-dimensional subspaces.

   Let P be a plane (i.e., a 2-dimensional subspace) of E. Assume that each element of P is isotropic. Then, from the identity $Q(u+v) = Q(u) + Q(v) + 2\varphi(u, v)$, it follows that any $u \in P$ belongs to $P^\perp$, hence, as $P^\perp$ is 1-dimensional, $Ku = P^\perp$, which is a contradiction. It follows that there exists at least one nonisotropic atom in P.

   Let Ku be a nonisotropic atom in P, and let v be a vector in P which is not colinear with u. For each atom $Kw \neq Ku$ in P, there exists a unique $\lambda \in K$ such that $Kw = K(\lambda u + v)$. This atom Kw is isotropic iff $Q(\lambda u + v) = \lambda^2 Q(u) + 2\lambda\varphi(u, v) + Q(v) = 0$. This equation has at most two solutions, hence there exists at most two isotropic atoms in P.
2. Let Ku be a nonisotropic atom in L. The map $P \mapsto P^\perp$ is one-to-one from the set of all nonisotropic planes containing Ku onto the set of nonisotropic atoms in $(Ku)^\perp$. It follows from 1. that there exist at least four nonisotropic planes containing the atom Ku.

□

Theorem 2.2.

1. $(T(E, \varphi), \subseteq, \perp)$ *is an orthomodular lattice.*
2. *The following are equivalent:*
   a) $T(E, \varphi)$ *is modular*
   b) $T(E, \varphi) = L(E, \varphi)$
   c) the bilinear form φ *is anisotropic* (*in other words, it admits none isotropic vector*).
3. *The orthomodular lattice* $T(E, \varphi)$ *is irreducible and* 3*-homogeneous* (*this means that each of its blocks has exactly* 3 *atoms*).

Proof:

1. J. Flachsmeyer (1995) has proved (in the more general case where E is any finite dimensional vector space over K), that $T(E, \varphi)$ is an orthomodular poset. However, let us give the proof in this particular case. We need only prove that if u, v are nonzero and nonisotropic vectors of E, $Ku \subset (Kv)^\perp$ implies $(Ku)^\perp \wedge (Kv)^\perp \neq 0$. This is equivalent to : $u \perp v$ implies that $Ku + Kv$ is not isotropic.

   Assume that $u \perp v$ and $Ku + Kv$ is isotropic. Then there exists an isotropic vector ω such that $\omega \perp u$ and $\omega \perp v$, hence both u and ω belong to $(Kv)^{\perp} \cap (K\omega)^{\perp}$ which is 1-dimensional. We obtain that u and ω are colinear, hence a contradiction since ω is isotropic and u is not isotropic.

2. It is obvious that b) implies a) and that b) and c) are equivalent.

   In order to prove that a) implies b), assume $T(E, \varphi) \neq L(E, \varphi)$, and let us prove that $T(E, \varphi)$ is not modular. There exists an isotropic vector ω in E. By the Lemma above, there exist two nonisotropic atoms Ku, Kv in the plane $(K\omega)^{\perp}$, and there exists a nonisotropic plane P such that $Ku \subseteq P$ and $Kv \nsubseteq P$. Then we have $Ku \subset P$, $(Ku \vee Kv) \wedge P = E \wedge P = P$, and $Ku \vee (Kv \wedge P) = Ku \vee \{0\} = Ku \neq P$, which shows that $T(E, \varphi)$ is not modular.

3. It follows from the previous Lemma that a 2-dimensional subspace of E is not an atom of $T(E, \varphi)$, hence any atom of $T(E, \varphi)$ is 1-dimensional. Now, let B be the set of all atoms of a block of $T(E, \varphi)$. Then, the elements of B are 1-dimensional subspaces of E, pairwise orthogonal, whose supremum in $T(E, \varphi)$ is E. By the part 1. of this proof, if Ku and Kv are two atoms of B, then $Ku + Kv$ is nonisotropic, hence $Ku \vee Kv = Ku + Kv$ is 2-dimensional. This shows that B cannot be 2-element and, as E is 3-dimensional, it follows that B is 3-element.

   As $T(E, \varphi)$ is of height 3, if $T(E, \varphi)$ is not irreducible, it is isomorphic to an orthomodular lattice of the form $T_1 \times T_2$ where T_1 and T_2 are of heights 1 and 2, hence are modular. It follows that $T(E, \varphi)$ is itself modular, hence φ is anisotropic, and it is well known that, in this classical case, $T(E, \varphi) = L(E, \varphi)$ is irreducible, which is a contradiction. □

3. ON THE SUB-ORTHOMODULAR LATTICES OF $T(E, \varphi)$

Let K' be a subfield of K. By definition, an orthogonal basis $e = (e_1, e_2, e_3)$ of E is said to be K'-closed if there exists $\alpha \in K$, $\alpha \neq 0$, such that, for $i = 1, 2, 3$, $\alpha Q(e_i) \in K'$.

Under these conditions, if E' is the K'-subspace of E (i.e., the linear subspace of E when E is regarded as a vector space over K') generated by the basis e, and φ' is the restriction to $E' \times E'$ of $\alpha\varphi$, then φ' is a nonsingular symmetric bilinear form over E', called the form induced by φ on E' (which is defined up to a constant factor in K').

Our main result is the following stucture Theorem about subalgebras of $T(E, \varphi)$.

Theorem 3.3.

1. *Direct part.*

   Let K' be a subfield of K and let $e = (e_1, e_2, e_3)$ be a K'-closed orthogonal basis of E.

Let E' be the K'-subspace of E generated by e and let φ' be the symmetric bilinear form induced by φ on E'.

Then $T(E', \varphi')$ is isomorphic to a subalgebra of $T(E, \varphi)$.
More precisely, the map which assigns to any M in $T(E', \varphi')$ the K-subspace of E generated by M is an injective homomorphism of orthomodular lattices from $T(E', \varphi')$ to $T(E, \varphi)$.
Moreover, we remark that $T(E', \varphi')$ is irreducible and 3-homogeneous.

2. *Converse part.*
Let T' be an irreducible and 3-homogeneous subalgebra of $T(E, \varphi)$.
Then there exist:
   - *a subfield K' of K,*
   - *a K'-closed orthogonal basis $e = (e_1, e_2, e_3)$ of E such that, if E' is the K'-subspace of E generated by e , and φ' the bilinear form induced by φ on E', then T' is isomorphic to $T(E', \varphi')$.*

Proof of the direct part: Let $\alpha \in K$, $\alpha \neq 0$, such that φ' is the restriction of $\alpha\varphi$ to $E' \times E'$, and let $h : L(E', \varphi') \mapsto L(E, \varphi)$ be the mapping which assigns to any $M \in L(E', \varphi')$ the K-subspace of E generated by M.

We notice that if M is the K'-subspace of E' generated by a list s of vectors, then $h(M)$ is the K-subspace of E generated by s. Now, let us suppose that s is a basis of M. Then s can be expanded to a basis $e' = (e'_1, e'_2, e'_3)$ of E'. The determinant of (e'_1, e'_2, e'_3) relative to the basis e of E' is nonzero, hence, as e is a basis of the K-space E, it follows that e' is also a basis of the K-space E. Thus, the vectors of s are linearly independent in E, hence s is a basis of the K-space $h(M)$. This proves that h preserves the dimension, and also that $M = E' \cap h(M)$, which shows that h is one-to-one.

Let us denote respectively by $\perp$ and $\perp'$ the polarities of $L(E, \varphi)$ and $L(E', \varphi')$. Let $K'u$ be any atom of $L(E', \varphi')$, and let (v, w) be a basis of $(K'u)^{\perp'}$. Then (v, w) is a basis of the K-space $h((K'u)^{\perp'})$. Since v, w belong to $(Ku)^{\perp}$, whose dimension is 2, and are independent vectors of E, (v, w) is a basis of $(Ku)^{\perp}$, and we conclude that $h((K'u)^{\perp'}) = (Ku)^{\perp} = (h(K'u))^{\perp}$. It follows easily that, for any $M \in L(E', \varphi')$, $h(M^{\perp'}) = (h(M))^{\perp}$.

An easy consequence of the definition of h is that, for any $M, N \in L(E', \varphi')$, $h(M + N) = h(M) + h(N)$, and, by the de Morgan laws, we infer that h is a lattice homomorphism from $L(E', \varphi')$ to $L(E, \varphi)$.

It is obvious that for any atom $K'u$ of $L(E', \varphi')$, $K'u$ is isotropic iff $h(K'u) = Ku$ is an isotropic atom of $L(E, \varphi)$. It follows that, for any $M \in L(E', \varphi')$, $h(M)$ is isotropic iff M is isotropic, and in particular that, for any $M \in T(E', \varphi')$, $h(M) \in T(E, \varphi)$.

Let g be the mapping from $T(E', \varphi')$ to $T(E, \varphi)$ defined by $g(M) = h(M)$.

If $M, N \in T(E', \varphi')$, then $M + N$ is an isotropic subspace of E' iff $h(M + N) = h(M) + h(N) = g(M) + g(N)$ is an isotropic subspace of E. It follows that $g(M \vee N) = g(M) \vee g(N)$ (the l.u.b. being taken resp. in $T(E', \varphi')$ and in $T(E, \varphi)$), and, by the de Morgan laws, that g is an (injective) homomorphism of orthomodular lattices from $T(E', \varphi')$ to $T(E, \varphi)$.

Remarks

1. The converse part of Theorem 2 is much more long and difficult to prove. In its proof we use:
   - the coordinatization Theorem for Arguesian projective planes,
   - the distance between an atom and a block of an orthomodular lattice,
   - the algebraic closure K^* of K and the embedding of the K-space E into a 3-dimensional K^*-space E^* equipped with a bilinear form φ^* inducing φ on E.
   - classical methods of projective geometry.
2. If T' satisfies the hypothesis of the converse part, the following sentences are equivalent:
   a) T' is a modular lattice
   b) the bilinear form φ' is anisotropic
   c) T' is a sublattice of $L(E, \varphi)$.
   For example, these conditions are satisfied in the case where K is the field $\mathbf{C}$ of complex numbers, $E = \mathbf{C}^3$, $\varphi((x, y, z), (x', y', z')) = xx' + yy' + zz'$, K' is the field of real numbers, $E' = \mathbf{R}^3$, and φ' is the restriction of φ to E'. In this case, φ is not anisotropic, thus $T(E, \varphi)$ is nonmodular, but φ' is anisotropic, hence $T(E', \varphi')$ is modular.
3. Theorem 2 does not work if the field K is of characteristic 3.
   Indeed, if $K = F_3$ (the 3-element field), and $E = F_3^3$ then $T(E, \varphi)$ does not depend (up to isomorphism) on the choice of the nonsingular bilinear form φ. The Greechie diagram of this orthomodular lattice is given in Fig. 1. The black atoms in this diagram, and the three blocks containing these atoms constitute the diagram of a proper, irreducible, 3-homogeneous

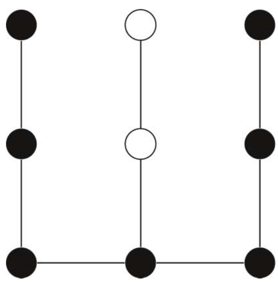

Fig. 1.

sub-orthomodular lattice of $T(E, \varphi)$, which is not associated to a subfield of K.

4. MINIMAL ORTHOMODULAR LATTICES

We recall that a nonmodular orthomodular lattice L is called minimal if all its proper subalgebras are either modular or isomorphic to L. If L is finite, this is equivalent to saying that all the proper subalgebras of L are modular. Recall that a main interest of minimality comes from the fact that a finite orthomodular lattice T is minimal if and only if the equational class generated by T covers an equational class of the form $[Mon]$, for some $n \geq 2$ (where $[Mon]$ denotes the equational class generated by the finite orthocomplemented modular lattice Mon).

Theorem 2 provides infinitely many finite minimal orthomodular lattices, and an infinite one.

1. Let us suppose that K is finite. It is well known that the cardinality q of K is of the form $q = p^n$, where p is a prime number. Here the conditions on the characteristic of K show that $p \neq 2$ and $p \neq 3$.

   If E is a 3-dimensional vector space over K, if φ_1, φ_2 are any two nonsingular forms on E, and if Q_1, Q_2 are respectively the corresponding quadratic forms, then there exists α in K such that Q_1 and αQ_2 are equivalent.

   This implies that orthomodular lattices $T(E, \varphi_1)$ and $T(E, \varphi_2)$ are isomorphic. Hence, up to isomorphism, the orthomodular lattice $T(E, \varphi)$ depends only on the cardinality of K.

   Moreover, it is easy to see that the previous result (concerning φ_1 and φ_2) allows us, for any subfield K' of K, to get a K'-closed orthogonal basis. It follows that the three following sentences are equivalent:
   a) $T(E, \varphi)$ is minimal,
   b) $n = 1$,
   c) q is a prime number.
   and that, up to isomorphism, the minimal orthomodular lattice $T(E, \varphi)$ depend only on the cardinality of K.

   So, we obtain, for each prime integer $p \geq 5$ a finite minimal orthomodular lattice T_p. This completes the study presented with Richard Greechie in Liptovsky Jan (Carrega *et al.*, 2000), where we had obtained these lattices only in the case where p is of the form $4k + 3$.
2. As concern the fields of characteristic 2, it is still possible to construct, by a slightly different way, the orthomodular lattice $T(E, \varphi)$, and we have already obtained in this way (Carrega (1998)) infinitely many finite minimal orthomodular lattices from finite fields of cardinal 2^p, where $p = 1$

or p is a prime number. We have in preparation a structure Theorem in characteristic 2, similar to Theorem 2 above.

3. If K is the field $\mathbf{Q}$ of rational numbers, and $E = \mathbf{Q}^3$, the orthomodular lattice $T(E, \varphi)$ does not depend (up to isomorphism) on the choice of the nonsingular and nonanisotropic form φ. This orthomodular lattice is infinite and, as $\mathbf{Q}$ is a prime field, $T(E, \varphi)$ is minimal.

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